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Introduction

Recently a number of writers have discussed interesting developments in the theory of not completely reducible matrix sets and non-semisimple algebras.¹ Here we have made use of some of these concepts and methods to study matrix algebras over an algebraically closed field. We shall discuss in some detail in this introduction the definitions that are used and the theorems that are developed.

Let $\mathfrak{A}$ denote a matrix algebra, with unit element, over an algebraically closed field K . $\mathfrak{A}$ will be taken in reduced form, by which we shall mean that $\mathfrak{A}$ is exhibited with only zeros above the main diagonal, with irreducible constituents of $\mathfrak{A}$ in the main diagonal, and that $\mathfrak{A}$ is expressible as a direct sum of its radical and a semisimple subalgebra which latter has non-zero components only in the irreducible constituents of $\mathfrak{A}$.²

$$(1) \quad \mathfrak{A} = \begin{pmatrix} \mathfrak{C}_{11} & & & \\ \mathfrak{C}_{21} & \mathfrak{C}_{22} & & \\ \vdots & & \ddots & \\ \vdots & & & \mathfrak{C}_{tt} \\ \mathfrak{C}_{t1} & \mathfrak{C}_{t2} & \cdots & \mathfrak{C}_{tt} \end{pmatrix},$$

the $\mathfrak{C}_{ii}$ denoting the irreducible constituents; further

$$\mathfrak{A} = \mathfrak{N} + \mathfrak{A}^*,$$

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¹ See, for instance, the forthcoming paper by R. Brauer, *On Sets of Matrices with Coefficients in a Division Ring*; T. Nakayama, *On Frobeniusean Algebras*, I, these Annals, Vol. 40 (1939), pp. 611-633; T. Nakayama, *Some Studies on Regular Representations, Induced Representations, and Modular Representations*, these Annals, Vol. 39 (1938), pp. 361-369; T. Nakayama and C. Nesbitt, *Note on Symmetric Algebras*, these Annals, Vol. 39 (1938), pp. 659-668; R. Brauer and C. Nesbitt, *On the Regular Representations of Algebras*, Proc. Nat. Acad. of Sci., Vol. 23 (1937), pp. 236-240; C. Nesbitt, *On the Regular Representations of Algebras*, these Annals, Vol. 39 (1938), pp. 634-658. We shall refer to the last two as B.N. and N.R., respectively.

² Cf. N.R., p. 639.

where $\mathfrak{N}$ is the radical of $\mathfrak{A}$ and

$$(2) \quad \mathfrak{N} = \begin{pmatrix} 0 & & & \\ \mathfrak{C}_{21} & 0 & & \\ \vdots & & \ddots & \\ \mathfrak{C}_{t1} & \mathfrak{C}_{t2} & \cdots & 0 \end{pmatrix}, \quad \mathfrak{A}^* = \begin{pmatrix} \mathfrak{C}_{11} & & & \\ 0 & \mathfrak{C}_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & \mathfrak{C}_{tt} \end{pmatrix}.$$

Let $F_1, F_2, \dots, F_k$ denote the totality of distinct irreducible representations of $\mathfrak{A}$. Then each $\mathfrak{C}_{ii}$ is equivalent to one of the F_k . It is well known that up to equivalence the irreducible constituents $\mathfrak{C}_{ii}$ of $\mathfrak{A}$ are uniquely determined.

As a part of $\mathfrak{A}$, $\mathfrak{C}_{ij}$ forms an additive group or module of matrices upon which $\mathfrak{A}$, itself considered as a module, is homomorphically mapped. We wish to consider $\mathfrak{C}_{ij}$ as a matrix module with $\mathfrak{A}$ as both a left and right operator system. For a matrix A of $\mathfrak{A}$, let us use the notation $C_{ij}(A)$, ($j \leq i, i = 1, 2, \dots, t$) to denote the parts of A ,

$$(3) \quad A = \begin{pmatrix} C_{11}(A) & & & \\ C_{21}(A) & C_{22}(A) & & \\ \vdots & \vdots & \ddots & \\ C_{t1}(A) & C_{t2}(A) & \cdots & C_{tt}(A) \end{pmatrix}.$$

Let B be any element of $\mathfrak{A}$, and let B^* be the component of B in the semisimple subalgebra $\mathfrak{A}^*$. We define B as a left and as a right operator of $C_{ij}(A)$ by the relations below, using $\circ$ to distinguish this operation from ordinary matrix multiplication

$$(4) \quad \begin{aligned} B \circ C_{ij}(A) &= C_{ii}(B) \cdot C_{ij}(A) = C_{ij}(B^*A), \\ C_{ij}(A) \circ B &= C_{ij}(A) \cdot C_{jj}(B) = C_{ij}(AB^*). \end{aligned}$$

Let us call a module which has $\mathfrak{A}$ as both right and left operator system an $(\mathfrak{A}, \mathfrak{A})$ module, and a homomorphism which is an operator mapping under the system of operators $\mathfrak{A}$, acting on both sides, an $(\mathfrak{A}, \mathfrak{A})$ homomorphism. Under definition (4), $\mathfrak{C}_{ij}$ is a simple $(\mathfrak{A}, \mathfrak{A})$ module. Moreover, each $\mathfrak{C}_{ij}$ is either a 0-part or there exist elements in $\mathfrak{A}$ such that the corresponding C_{ij} have any arbitrary components from K .

Accordingly, we shall call the $\mathfrak{C}_{ij}$ the *simple parts* of $\mathfrak{A}$. The simple parts $\mathfrak{C}_{ii}$, or irreducible constituents of $\mathfrak{A}$, have been rather thoroughly studied. Our first aim, then, is to develop a theory of simple parts which includes the $\mathfrak{C}_{ij}$, with $i \neq j$, that is, the simple parts of $\mathfrak{A}$ which appear in the radical $\mathfrak{N}$ of $\mathfrak{A}$.

A first remark is that, when $\mathfrak{A}$ is considered as an $(\mathfrak{A}, \mathfrak{A})$ module, it can be proved in a direct manner that each $\mathfrak{C}_{ij}$ which is not a 0-part is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to a composition factor group of $\mathfrak{A}$. This result may also be obtained by applying a useful theorem given recently by R. Brauer.³

³ R. Brauer, op. cit.

When $\mathfrak{C}_{ii}$, $\mathfrak{C}_{jj}$ are respectively the irreducible constituents F_κ , F_λ of $\mathfrak{A}$, we shall say that $\mathfrak{C}_{ij}$ is of type (κ, λ) . If $\kappa \neq \lambda$, we shall say that $\mathfrak{C}_{ij}$ is of mixed type, and if $\kappa = \lambda$ that $\mathfrak{C}_{ij}$ is of unmixed type. An $(\mathfrak{A}, \mathfrak{A})$ isomorphism may be defined for any two $\mathfrak{C}_{ij}$ of type (κ, λ) and, as a consequence, the corresponding factor groups of $\mathfrak{A}$ are also isomorphic. It is easy to see that two simple parts not of the same type are not $(\mathfrak{A}, \mathfrak{A})$ isomorphic. A representation $\mathfrak{A}$ of $\mathfrak{A}$ may be considered as an $(\mathfrak{A}, \mathfrak{A})$ module. Again applying Brauer's Theorem it follows that the simple parts of type (κ, λ) of a representation $\mathfrak{A}$, which is in reduced form, are $(\mathfrak{A}, \mathfrak{A})$ isomorphic to the simple parts of $\mathfrak{A}$ of type (κ, λ) .

In N.R. extensive use was made of a basis system of $\mathfrak{A}$ first employed by Cartan⁴, and which was there called the Cartan basis system. The coefficients in the expressions for the elements A of $\mathfrak{A}$ as linear combinations of the Cartan basis elements yield a set of matrix modules $\mathfrak{S}^\mu$, ($\mu = 1, 2, \dots, m$) upon which $\mathfrak{A}$, as a module, is homomorphically mapped. The $\mathfrak{S}^\mu$ may also be considered as $(\mathfrak{A}, \mathfrak{A})$ modules. We shall call these $\mathfrak{S}^\mu$ *elementary (related) modules* of $\mathfrak{A}$. The number m of elementary modules is equal to the composition length of $\mathfrak{A}$ considered as an $(\mathfrak{A}, \mathfrak{A})$ module, and each $\mathfrak{S}^\mu$ is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to a composition factor group of $\mathfrak{A}$. Conversely, to each composition factor group of a composition series of $\mathfrak{A}$ there corresponds an isomorphic elementary module. It follows that up to $(\mathfrak{A}, \mathfrak{A})$ isomorphism the set of elementary modules is uniquely determined.

We say that an elementary module $\mathfrak{S}^\mu$ is of type (κ, λ) if $\mathfrak{S}^\mu$ is defined by Cartan basis elements of type (κ, λ) . Simple parts and elementary modules may also be classified by the powers of the radical to which they belong. A simple part $\mathfrak{C}_{ij}$ (an elementary module $\mathfrak{S}^\mu$) *belongs to* $\mathfrak{N}^h$ if this is the highest power of the radical such that all its elements are not mapped on the 0-matrix in $\mathfrak{C}_{ij}$ ($\mathfrak{S}^\mu$). The principal relation between simple parts and elementary modules is that every simple part $\mathfrak{C}_{ij}$ of type (κ, λ) which belongs to $\mathfrak{N}^h$ is expressible as a linear combination of elementary modules $\mathfrak{S}^\mu$ of type (κ, λ) which belong to $\mathfrak{N}^j$, $j \leq h$.

The form of a center element of $\mathfrak{A}$ can be computed. For a center element Z , $C_{ij}(Z) = 0$ if $\mathfrak{C}_{ij}$ is a simple part of mixed type, and $C_{ij}(Z)$ is of form $(c \cdot \delta_{mn})$, c and element of K , if $\mathfrak{C}_{ij}$ is of unmixed type. Let $c_{\kappa\lambda}$ denote, as in B.N., the Cartan Invariants. Then the rank ρ of the center is at most equal to the number $s = \sum_{\kappa=1}^h c_{\kappa\kappa}$ of elementary modules of unmixed type.

As an application of these concepts we obtain a decomposition of the matrix algebra $\mathfrak{A}$. By use of the concept of type, the simple parts of $\mathfrak{A}$ may be classified into 'blocks'. If $\mathfrak{C}_{ii}$, $\mathfrak{C}_{jj}$, $j < i$, do not belong to the same block then $\mathfrak{C}_{ij}$ must be a 0-part. An immediate consequence is that $\mathfrak{A}$ may be decomposed into parts which contain all simple parts belonging to a block. Each such part defines an invariant subalgebra which may not be decomposed into a direct sum of invariant subalgebras.

⁴ E. Cartan, *Les groupes bilinéaires et les systèmes de nombres complexes*, Annales de Toulouse, 12B (1898), p. 1.

1

We consider a matrix algebra $\mathfrak{A}$, with unit element E , over an algebraically closed field K . We may assume that $\mathfrak{A}$ has no 0-constituent.⁵ We take $\mathfrak{A}$ in the reduced form (1) and define operators for the parts $\mathfrak{C}_{ij}$ of $\mathfrak{A}$ by (4). The field K may be considered as included in the operator system by identifying the element k of K with the element $k \cdot E$ of $\mathfrak{A}$.

By means of (4) the parts $\mathfrak{C}_{ij}$ of $\mathfrak{A}$ (cf. (1)) may be regarded as $(\mathfrak{A}, \mathfrak{A})$ modules. If $\mathfrak{C}_{ij}$ contains only the 0-matrix, we call $\mathfrak{C}_{ij}$ a 0-part, and obviously, in this case, $\mathfrak{C}_{ij}$ is a simple module. Suppose, instead, that there exists an element A of $\mathfrak{A}$ such that $C_{ij}(A) \neq 0$. It is well known that elements L of $\mathfrak{A}$ can be chosen so that the corresponding $C_{ii}(L)$, $C_{jj}(L)$ have any components arbitrarily chosen from K . It follows from (4) that by suitably choosing right and left operators L , M , $\dots$ to be applied to $C_{ij}(A)$ one obtains a system of matrices

$$C_{ij}(L^*AM^*) = C_{ii}(L)C_{ij}(A)C_{jj}(M)$$

in $\mathfrak{C}_{ij}$, from which may be generated matrices of $\mathfrak{C}_{ij}$ with arbitrary coefficients. Thus, the admissible subgroup which contains $C_{ij}(A)$ is the whole module $\mathfrak{C}_{ij}$. Let us call a module of matrices, each of m rows and n columns and with coefficients in K , a *complete module* if it contains all such matrices. We then have

THEOREM 1: *Each $\mathfrak{C}_{ij}$ is either a 0-part or a complete module. In either case $\mathfrak{C}_{ij}$ is a simple $(\mathfrak{A}, \mathfrak{A})$ module.*

We shall let $\mathfrak{N}^r$ denote the set which contains all elements of the form $\sum N_{\mu_1}N_{\mu_2} \cdots N_{\mu_r}$, where the N_{μ_i} are elements contained in the radical $\mathfrak{N}$. For a sufficiently large value of t , $\mathfrak{N}^t = 0$. These $\mathfrak{N}^r$ form invariant subalgebras of $\mathfrak{A}$. If we denote $\mathfrak{A}$ by $\mathfrak{N}^0$, we have

$$\mathfrak{A} = \mathfrak{N}^0 \supset \mathfrak{N}^1 \supset \mathfrak{N}^2 \supset \mathfrak{N}^3 \supset \cdots \supset \mathfrak{N}^{l-1} \supset \mathfrak{N}^l = 0.$$

We say that an element A belongs to $\mathfrak{N}^h$ if h is the largest value such that $\mathfrak{N}^h$ contains A . In particular, the elements of $\mathfrak{A}^*$ belong to $\mathfrak{N}^0$. The product of an element belonging to $\mathfrak{N}^h$ with an element belonging to $\mathfrak{N}^j$ belongs to $\mathfrak{N}^k$ with $k \geq h + j$.

We can now prove

THEOREM 2: *A non-zero simple part $\mathfrak{C}_{ij}$ of $\mathfrak{A}$ is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to a composition factor group of $\mathfrak{A}$ itself considered as an $(\mathfrak{A}, \mathfrak{A})$ module.*

PROOF: Take the series

$$\mathfrak{A} \supset \mathfrak{N}^1 \supset \mathfrak{N}^2 \supset \mathfrak{N}^3 \supset \cdots \supset \mathfrak{N}^{l-1} \supset \mathfrak{N}^l = 0$$

and refine it to obtain a composition series

$$\mathfrak{A} \supset \cdots \supset \mathfrak{A}_{q-1} \supset \mathfrak{A}_q \supset \cdots \supset \mathfrak{A}_{n-1} \supset 0$$

⁵ For, from conditions stated for $\mathfrak{A}$, we may show that if $\mathfrak{A}$ has 0-constituents, then it may be decomposed in the form $\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\mathfrak{A}_1$ does not have 0-constituents, and $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_1$.

of $\mathfrak{A}$. Let $\mathfrak{A}_q$ be the first group in this series such that for each of its elements A , $C_{ij}(A) = 0$. Then denoting elements of $\mathfrak{A}_{q-1}$ by A_{q-1} , we have that

$$A_{q-1} \rightarrow C_{ij}(A_{q-1})$$

is an $(\mathfrak{A}, \mathfrak{A})$ homomorphism, since for any element of $\mathfrak{A}$,

$$\begin{aligned} B \cdot A_{q-1} &= B^* \cdot A_{q-1} + N \cdot A_{q-1} \\ &\rightarrow C_{ij}(B^* A_{q-1}) + C_{ij}(N \cdot A_{q-1}). \end{aligned}$$

But $N \cdot A_{q-1} \subset \mathfrak{A}_q$ so that $C_{ij}(N A_{q-1}) = 0$ and, therefore,

$$B \cdot A_{q-1} \rightarrow C_{ij}(B^* A_{q-1}) = B \circ C_{ij}(A_{q-1}).$$

The same argument holds for right side multiplication. The kernel of this mapping (that is, the elements mapping into the identity element) contains $\mathfrak{A}_q$ and, therefore, the kernel is $\mathfrak{A}_q$. It follows that $\mathfrak{A}_{q-1}/\mathfrak{A}_q$ is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to the simple part $\mathfrak{C}_{ij}$.

We have said that if $\mathfrak{C}_{ii}$, $\mathfrak{C}_{jj}$ are respectively the irreducible constituents F_κ , F_λ of $\mathfrak{A}$, then the simple part $\mathfrak{C}_{ij}$ is of type (κ, λ) . We now show that

THEOREM 3: *The non-zero simple parts $\mathfrak{C}_{ij}$ of a given type (κ, λ) are mutually isomorphic in the $(\mathfrak{A}, \mathfrak{A})$ sense. Simple parts of different types are not isomorphic.*

PROOF: It should be noted here that the word *isomorphic* is to have a meaning differing from that of Theorem 2. In Theorem 2, the isomorphism is obtained from $A \rightarrow C_{ij}(A)$, that is, $\mathfrak{C}_{ij}$ is considered as a set of matrices related to $\mathfrak{A}$. Here, $\mathfrak{C}_{ij}$ is just to represent the system of matrices.

Let $\mathfrak{C}_{ij}$, $\mathfrak{C}_{mn}$ be two simple parts of type (κ, λ) which are not 0-parts. An $(\mathfrak{A}, \mathfrak{A})$ mapping of $\mathfrak{C}_{ij}$ upon $\mathfrak{C}_{mn}$ is obtained by the relation $C_{ij} \rightarrow C_{mn}$ if $C_{ij} = C_{mn}$ where C_{ij} , C_{mn} denote matrices of $\mathfrak{C}_{ij}$, $\mathfrak{C}_{mn}$, respectively. For if B is any element of $\mathfrak{A}$, then

$$\begin{aligned} B \circ C_{ij} &= C_{ii}(B) \cdot C_{ij} = F_\kappa(B) \cdot C_{ij} \\ &= C_{mm}(B) \cdot C_{mn} = B \circ C_{mn}. \end{aligned}$$

On the other hand, if $\mathfrak{C}_{ij}$, $\mathfrak{C}_{mn}$ are of types (κ, λ) , (μ, ν) , respectively, with $\kappa \neq \mu$, then any relation $C_{ij} \rightarrow C_{mn}$ is not an $(\mathfrak{A}, \mathfrak{A})$ operator mapping, for we may choose B in $\mathfrak{A}$ such that $C_{ii}(B) = F_\kappa(B) = 0$ while $C_{mm}(B) = F_\mu(B) = E_{f_\mu}$ where E_{f_μ} is a unit matrix of proper degree.

2

The basis of $\mathfrak{A}$ which seems best adapted to the study of simple parts is a basis system first employed by Cartan.⁶ This basis was used by C. Nesbitt in his study of regular representations of linear associative algebras.⁷

We denote, as before, the irreducible representations of $\mathfrak{A}$ by F_1 , F_2 , F_3 ,

⁶ E. Cartan, op. cit.

⁷ A detailed description of the Cartan basis system is given in N.R. section 2.

$\cdots, F_k$, and their degrees by $f_1, f_2, \cdots, f_k$. Let $E_\kappa(m, n)$ be the matrix of $\mathfrak{A}$ which has 1 for the (m, n) component of each $\mathfrak{C}_{ii}$ which is equal to F_κ , and has zero elsewhere. The relations

$$E_\mu(m, n) \cdot E_\nu(p, q) = \begin{cases} 0 & \text{if either } \mu \neq \nu, \text{ or } p \neq q, \\ E_\mu(m, q) & \text{if } \mu = \nu, n = p, \end{cases}$$

follow from the definition of the $E_\mu(m, n)$.

Let $E_\kappa = \sum_i E_\kappa(ii)$. Then the E_κ , ($\kappa = 1, 2, \cdots, k$), form a system of mutually idempotent elements. In fact, E_κ is the unit element of a certain simple algebra which is a summand in a decomposition of $\mathfrak{A}^*$ into a direct sum of simple algebras, and $\sum_{\kappa=1}^k E_\kappa = E$, where E is the unit element of $\mathfrak{A}$.

A matrix A of $\mathfrak{A}$ is said to be of type (κ, λ) if $E_\kappa \cdot A \cdot E_\lambda = A$. In particular, $E_\kappa(m, n)$ is of type (κ, κ) . An element of type (κ, λ) , $\kappa \neq \lambda$, we shall say is of mixed type, and if $\kappa = \lambda$, of unmixed type. The product of an element of type (κ, λ) and an element of type (μ, ν) is an element of type (κ, ν) . The product of an element of type (κ, λ) and of an element of type (μ, ν) vanishes for $\lambda \neq \mu$. A matrix A of type (κ, λ) has non-zero components only in simple parts $\mathfrak{C}_{ij}$ of type (κ, λ) .

A basis set $B_{\kappa\lambda}^1, B_{\kappa\lambda}^2, \cdots, B_{\kappa\lambda}^t$ is first obtained for those matrices of $\mathfrak{A}$ of type (κ, λ) which have non-zero components only in the upper left corners of simple parts of type (κ, λ) . The rank t of this set is the Cartan Invariant, $c_{\kappa\lambda}$.⁸ A basis for the whole algebra is then given by the set

$$\begin{aligned} & E_\kappa(a1)B_{\kappa\lambda}^\mu E_\lambda(1b), \\ & \mu = 1, 2, \cdots, c_{\kappa\lambda}, \quad a = 1, 2, \cdots, f_\kappa, \\ & b = 1, 2, \cdots, f_\lambda, \quad \kappa, \lambda = 1, 2, \cdots, k. \end{aligned}$$

This basis can be so chosen that any element A of $\mathfrak{A}$ which belongs to $\mathfrak{N}^\tau$ is expressible in terms of basis elements which belong to $\mathfrak{N}^\rho$, for $\rho \geq \tau$.⁹

For convenience we change slightly the above notation. We take together in one set all the elements $B_{\kappa\lambda}^\mu$ for all κ, λ , and μ and redefine the superscript μ to enumerate these elements, independently of type, in such a way as to take first all those elements which belong to $\mathfrak{N}^0$, namely $E_1(11), \cdots, E_k(11)$, then next those which belong to $\mathfrak{N}^1$, and so on. μ will now have the range $1, 2, \cdots, m$ where $m = \sum_{\kappa, \lambda=1}^k c_{\kappa\lambda}$. Where the type of the element need not be explicitly stated we shall drop the subscripts κ, λ from the symbols $B_{\kappa\lambda}^\mu$.

3

Expressing the element A of $\mathfrak{A}$ as a linear combination of the elements of the Cartan basis system, we have

$$(5) \quad A = \sum_{a,b,\mu} h_{ab}^\mu(A) E_\kappa(a1) B_{\kappa\lambda}^\mu E_\lambda(1b).$$

⁸ See B.N. or N.R.

⁹ See N.R., p. 641.

Here each $B_{\kappa\lambda}^\mu$ is a basis element having components different from zero only in the upper left hand corners of simple parts $\mathfrak{S}_{ij}$ which are of type (κ, λ) .

We denote the matrix $(h_{ab}^\mu(A))_{ab}$ (cf. (5)) by $H^\mu(A)$. The module consisting of the matrices $H^\mu(A)$ will be denoted by $\mathfrak{S}^\mu$. We shall call $\mathfrak{S}^\mu$ an *elementary (related) module* of $\mathfrak{A}$. $\mathfrak{S}^\mu$ is evidently a complete module, since the coefficients in (5) may be taken arbitrarily from K .

An elementary module will be said to be of type (κ, λ) if it results from, or is defined by, Cartan basis elements of type (κ, λ) . We shall write $\mathfrak{S}_{\kappa\lambda}^\mu$ to denote that $\mathfrak{S}^\mu$ is of type (κ, λ) . If $\kappa \neq \lambda$ we shall call $\mathfrak{S}_{\kappa\lambda}^\mu$ an elementary module of *mixed type*, and if $\kappa = \lambda$ we shall call $\mathfrak{S}_{\kappa\lambda}^\mu$ an elementary module of *unmixed type*.

Let us say that a module $\mathfrak{M}$ of matrices is related to $\mathfrak{N}^\rho$ if $\mathfrak{M}$ is homomorphically mapped on $\mathfrak{N}$. In general, we shall deal with modules $\mathfrak{M}$ which are related to $\mathfrak{N}^0 = \mathfrak{A}$, but it is convenient sometimes to consider $\mathfrak{M}$ as the map of only the subalgebra $\mathfrak{N}^\rho$ rather than $\mathfrak{A}$ itself (see p. 157). A related module $\mathfrak{M}$ will be said to belong to $\mathfrak{N}^\sigma$ if σ is the largest value such that for at least one element N_σ belonging to $\mathfrak{N}^\sigma$, the corresponding matrix $M(N_\sigma)$ of $\mathfrak{M}$ is not zero. In particular, the elementary module $\mathfrak{S}^\mu$ is related to $\mathfrak{A}$, and belongs to the same power of $\mathfrak{N}$ as does the corresponding basis element B^μ .

LEMMA 1: Let $\mathfrak{M}$ be a module for which the concept of type is defined. If

- i). $\mathfrak{M}$ is of type (κ, λ) , and its elements are $f_\kappa \times f_\lambda$ matrices,
- ii). $\mathfrak{M}$ is related to $\mathfrak{N}^\rho$, and belongs to $\mathfrak{N}^\sigma$, and
- iii). $\mathfrak{M}$ is a simple $(\mathfrak{A}, \mathfrak{A})$ module such that for any element B of $\mathfrak{A}$, and element N_ρ of $\mathfrak{N}^\rho$

$$(6) \quad \begin{aligned} B \circ M(N_\rho) &= F_\kappa(B) \cdot M(N_\rho) = M(B^* N_\rho) \\ M(N_\rho) \circ B &= M(N_\rho) \cdot F_\lambda(B) = M(N_\rho B^*), \end{aligned}$$

then it follows that as a module related to $\mathfrak{N}^\rho$, $M(N_\rho) = \sum_\mu m_\mu H_{\kappa\lambda}^\mu(N_\rho)$, where m_μ are fixed scalars, N_ρ is any element of $\mathfrak{N}^\rho$, and where the elementary module $\mathfrak{S}_{\kappa\lambda}^\mu$ belongs to $\mathfrak{N}^\tau$, $\rho \leq \tau \leq \sigma$, and is considered as related to $\mathfrak{N}^\sigma$.

PROOF: We shall use the notation $(U_{rs})_{pq}$ to denote a matrix with 1 as its (r, s) component, and with zero elsewhere. Then operating with $E_\kappa(1, 1)$, $E_\lambda(1, 1)$ on $M(B_{\kappa\lambda}^\mu) = (m_{pq}^\mu)$, we obtain

$$\begin{aligned} E_\kappa(11) \circ M(B_{\kappa\lambda}^\mu) \circ E_\lambda(11) &= (U_{11})_{rs} \cdot (m_{pq}^\mu) \cdot (U_{11})_{uv} = (m_{11}^\mu) \cdot (U_{11})_{pq} \\ &= M(E_\kappa(11) B_{\kappa\lambda}^\mu E_\lambda(11)) = M(B_{\kappa\lambda}^\mu). \end{aligned}$$

Dropping the subscripts 11 we have

$$M(B_{\kappa\lambda}^\mu) = (m^\mu U_{11})_{pq}.$$

Here $m^\mu \neq 0$ only if $B_{\kappa\lambda}^\mu$ belongs to $\mathfrak{N}^\tau$, $\tau \leq \sigma$. Also

$$M(E_\kappa(a1) B_{\kappa\lambda}^\mu E_\lambda(1b)) = (m^\mu U_{ab})_{pq}.$$

We have finally that

$$\begin{aligned} M(N_\rho) &= M\left(\sum_{\mu, a, b} h_{ab}^\mu(N_\rho) E_\kappa(a1) E_{\kappa\lambda}^\mu E_\lambda(1b)\right) \\ &= \sum_\mu m^\mu H_{\kappa\lambda}^\mu(N^\rho) \end{aligned}$$

where $\mathfrak{S}_{\kappa\lambda}^\mu$ belongs to $\mathfrak{N}^\tau$, $\rho \leq \tau \leq \sigma$, since N_ρ is an element of $\mathfrak{N}^\rho$, and m belongs to $\mathfrak{N}^\sigma$.

Relating the simple parts of $\mathfrak{A}$ to the elementary modules of $\mathfrak{A}$, we have

THEOREM 4: *A simple part $\mathfrak{S}_{ij}$ of $\mathfrak{A}$ of type (κ, λ) which belongs to $\mathfrak{N}^\tau$ is expressible as a linear combination of the elementary modules of type (κ, λ) which belong to $\mathfrak{N}^\tau$, $\tau \leq r$.*

PROOF: We consider $\mathfrak{S}_{ij}$ as a simple module related to $\mathfrak{A} = \mathfrak{N}^0$, and belonging to $\mathfrak{N}^\tau$, and apply the lemma. It follows that $\mathfrak{S}_{ij}$ is of the form $\sum_\mu d_{ij}^\mu H_{\kappa\lambda}^\mu$, that is, for each element A of $\mathfrak{A}$ we have that

$$C_{ij}(A) = \sum_\mu d_{ij}^\mu H_{\kappa\lambda}^\mu(A),$$

where each $H_{\kappa\lambda}^\mu(A)$ belongs to some N^τ , $\tau \leq r$.

We now prove

THEOREM 5: *The number m of elementary modules is equal to the composition length of $\mathfrak{A}$ considered as an $(\mathfrak{A}, \mathfrak{A})$ module. Each elementary module $\mathfrak{S}$ is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to a composition factor group of $\mathfrak{A}$, and conversely, to each factor group of $\mathfrak{A}$ there corresponds an isomorphic elementary module.*

PROOF: We will construct a composition series for $\mathfrak{A}$ in terms of the Cartan basis elements. As indicated above, we take the whole set of elements $B_{\kappa\lambda}^\mu$ ($\kappa, \lambda = 1, 2, 3, \dots, k$) and use the superscript μ ($\mu = 1, 2, 3, \dots, m$) to enumerate these elements, taking first those elements which are not in $\mathfrak{N}^1$, then those which are in $\mathfrak{N}^1$ but not in $\mathfrak{N}^2$, and so on. Let $\mathfrak{A}_{s-1}$ denote the subgroup of $\mathfrak{A}$ consisting of all linear combinations with coefficients in K of the elements

$$\begin{aligned} &(\kappa, \lambda = 1, 2, \dots, \text{or } k), \\ (7) \quad &E_\kappa(a1) B_{\kappa\lambda}^\tau E_\lambda(1b), \quad (a = 1, 2, \dots, f_\kappa; \quad b = 1, 2, \dots, f_\lambda), \\ &(\tau = S, S+1, \dots, m). \end{aligned}$$

In particular, $\mathfrak{A}_0 = \mathfrak{A}$. We form the product

$$A \cdot E_\kappa(a1) B_{\kappa\lambda}^\tau E_\lambda(1b) = (A^* + N) \cdot E_\kappa(a1) B_{\kappa\lambda}^\tau E_\lambda(1b).$$

Here

$$\begin{aligned} A^* \cdot E_\kappa(a1) B_{\kappa\lambda}^\tau E_\lambda(1b) &= \sum_{\rho, m, n} f_{mn}^\rho(A) E_\rho(mn) \cdot E_\kappa(a1) B_{\kappa\lambda}^\tau E_\lambda(1b) \\ &= \sum_m f_{ma}^\kappa(A) E_\kappa(m1) B_{\kappa\lambda}^\tau E_\lambda(1b) \end{aligned}$$

is again in $\mathfrak{A}_{s-1}$. So also is $N \cdot E_\kappa(a1) B_{\kappa\lambda}^\tau E_\lambda(1b)$, since this belongs to a higher

power of the radical than does $B_{\kappa\lambda}^r$ and is, therefore, expressible in terms of the elements (7). Then the whole product is in $\mathfrak{A}_{s-1}$. The same is true when A is a right factor, so we obtain that $\mathfrak{A}_{s-1}$ is an admissible subgroup of $\mathfrak{A}$. Moreover, the factor group $\mathfrak{A}_{s-1}/\mathfrak{A}_s$ consisting of the residue classes

$$\langle \sum_{a,b} \lambda_{ab}^s E_{\kappa}(a1) B_{\kappa\lambda}^s E_{\lambda}(1b) \rangle, \quad \text{modulo } \mathfrak{A}_s,$$

is easily seen by similar calculations to be simple. It follows that

$$\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_{m-1} \supset 0$$

is a composition series of $\mathfrak{A}$, and the first part of the theorem is proved.

Again, we may easily calculate that

$$\langle \sum_{a,b} h_{ab}^s(A) E_{\kappa}(a1) B_{\kappa\lambda}^s E_{\lambda}(1b) \rangle \rightarrow H_{\kappa\lambda}^s(A)$$

is an $(\mathfrak{A}, \mathfrak{A})$ isomorphic mapping of $\mathfrak{A}_{s-1}/\mathfrak{A}_s$ upon the elementary module $\mathfrak{E}_{\kappa\lambda}^s$ (cf. (7)). From this follow the other statements in the theorem.

It may be noted here that from Theorem 4 and Theorem 5 another proof of Theorem 2 may be obtained.

We can now prove the converse of Theorem 2

THEOREM 6: *To each composition factor group of $\mathfrak{A}$ there corresponds at least one non-zero simple part $\mathfrak{E}_{ij}$.*

PROOF: By Theorem 5 we have that to each composition factor group of $\mathfrak{A}$ there corresponds an isomorphic elementary module $\mathfrak{E}^p$, say of type (μ, ν) . Then there exist non-zero simple parts of type (μ, ν) , and by the same argument as that used in Theorem 3 it may be shown that each non-zero simple part of type (μ, ν) is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to $\mathfrak{E}^p$.

4

A system of matrices $\bar{\mathfrak{A}}$ is called a representation of $\mathfrak{A}$ if $\mathfrak{A}$ is homomorphically mapped on $\bar{\mathfrak{A}}$. We may define $\mathfrak{A}$ as a left and right operator system for a representation $\bar{\mathfrak{A}}$ by the relations

$$(8) \quad \begin{aligned} A \cdot \bar{A}(A_1) &= \bar{A}(A \cdot A_1) = \bar{A}(A) \cdot \bar{A}(A_1), \\ \bar{A}(A_1) \cdot A &= \bar{A}(A_1 \cdot A) = \bar{A}(A_1) \cdot \bar{A}(A). \end{aligned}$$

Further, if the representation $\bar{\mathfrak{A}}$ is taken in reduced form,¹⁰ then a simple part $\bar{\mathfrak{E}}_{ij}$ of $\bar{\mathfrak{A}}$ may be considered as an $(\mathfrak{A}, \mathfrak{A})$ module if we define the elements of $\mathfrak{A}$ as operators in $\bar{\mathfrak{E}}_{ij}$ by relations of the form (4). We then obtain

THEOREM 7: *The simple parts $\bar{\mathfrak{E}}_{ij}$, of type (κ, λ) , of a representation $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ are $(\mathfrak{A}, \mathfrak{A})$ isomorphic¹⁰ to simple parts $\mathfrak{E}_{mn}$, of type (κ, λ) , of $\mathfrak{A}$.*

¹⁰ As in Theorem 3, the word *isomorphism* here has a somewhat different meaning from that of Theorem 2.

PROOF: By Theorem 2 we have that each simple part $\bar{\mathfrak{C}}_{ij}$ of $\bar{\mathfrak{A}}$ is isomorphic to a composition factor group of a composition series for $\bar{\mathfrak{A}}$. Let

$$(9) \quad \bar{\mathfrak{A}} \supset \bar{\mathfrak{A}}_1 \supset \bar{\mathfrak{A}}_2 \supset \cdots \supset \bar{\mathfrak{A}}_l = 0,$$

$$(10) \quad \mathfrak{A} \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_n = 0,$$

be composition series for $\bar{\mathfrak{A}}$, $\mathfrak{A}$, respectively, and suppose

$$\bar{\mathfrak{C}}_{ij} \cong \bar{\mathfrak{A}}_{p-1}/\bar{\mathfrak{A}}_p.$$

$\bar{\mathfrak{A}}$ is an $(\mathfrak{A}, \mathfrak{A})$ module upon which $\mathfrak{A}$ is homomorphically mapped by the relation $A \rightarrow \bar{A}(A)$. Then the Jordan-Hölder Theorem gives us that to the factor group $\bar{\mathfrak{A}}_{p-1}/\bar{\mathfrak{A}}_p$ of (9) there corresponds a factor group $\mathfrak{A}_{q-1}/\mathfrak{A}_q$ of the series (10) such that

$$\bar{\mathfrak{A}}_{p-1}/\bar{\mathfrak{A}}_p \cong \mathfrak{A}_{q-1}/\mathfrak{A}_q.$$

In addition we have from Theorem 6 that to the factor group $\mathfrak{A}_{q-1}/\mathfrak{A}_q$ there is at least one non-zero isomorphic simple part $\mathfrak{C}_{mn}$ of $\mathfrak{A}$,

$$\mathfrak{A}_{q-1}/\mathfrak{A}_q \cong \mathfrak{C}_{mn}.$$

Combining these relations, and observing that all these isomorphisms are $(\mathfrak{A}, \mathfrak{A})$ isomorphisms, we obtain

$$\bar{\mathfrak{C}}_{ij} \cong \mathfrak{C}_{mn}$$

with operators $(\mathfrak{A}, \mathfrak{A})$. This completes the proof of the statement.

Now suppose that $\bar{\mathfrak{A}}$ is a representation of $\mathfrak{A}$, and that $\bar{\mathfrak{A}}$ is in reduced form. Let us denote the element of $\bar{\mathfrak{A}}$ corresponding to the element A of $\mathfrak{A}$ by $\bar{A}$, $A \rightarrow \bar{A}$; further, we shall consider $\bar{\mathfrak{A}}$ as a direct sum of its radical $\bar{\mathfrak{A}}_R$ and a semi-simple subalgebra, $\bar{\mathfrak{A}}_S$, $\bar{\mathfrak{A}} = \bar{\mathfrak{A}}_S + \bar{\mathfrak{A}}_R$, where $\bar{\mathfrak{A}}_S$ is completely decomposed into its irreducible constituents. We use $\bar{A}_S$, $\bar{A}_R$ to denote the components in $\bar{\mathfrak{A}}_S$, $\bar{\mathfrak{A}}_R$ of the element $\bar{A}$ of $\bar{\mathfrak{A}}$, $\bar{A} = \bar{A}_S + \bar{A}_R$.

The set $\bar{\mathfrak{A}}^*$ of elements in the representation $\bar{\mathfrak{A}}$ which correspond to the semi-simple subalgebra $\mathfrak{A}^*$ of $\mathfrak{A}$ form a semisimple subalgebra of $\bar{\mathfrak{A}}$ equivalent to $\bar{\mathfrak{A}}_S$. Since $\bar{\mathfrak{A}}^*$, $\bar{\mathfrak{A}}_S$ have the same irreducible constituents, they are equivalent algebras, but are not necessarily identical. It is easy to construct examples where $\bar{\mathfrak{A}}^*$ is not $\bar{\mathfrak{A}}_S$.

Let us suppose $\bar{\mathfrak{S}}_1, \dots, \bar{\mathfrak{S}}_\theta$ are a set of elementary modules for the algebra $\bar{\mathfrak{A}}$. We wish to give explicitly the relation between the modules $\bar{\mathfrak{S}}$ of $\bar{\mathfrak{A}}$, and the elementary modules $\mathfrak{S}$ of $\mathfrak{A}$.

The irreducible constituents of $\bar{\mathfrak{A}}$ are equivalent to irreducible constituents of $\mathfrak{A}$. For simplicity, let us suppose they are identical with constituents of $\mathfrak{A}$. Let $\bar{\mathfrak{S}}$ be of type (κ, λ) and belong to the σ^{th} power of the radical of $\bar{\mathfrak{A}}$. We may consider $\bar{\mathfrak{S}}$ as a module related to $\mathfrak{A}$ by setting $\bar{H}(A) = \bar{H}(\bar{A})$. As such, $\bar{\mathfrak{S}}$ belongs to $\mathfrak{N}^\sigma$. Let B be any element of $\mathfrak{A}$, and N_σ an element of $\mathfrak{N}^\sigma$. Then

$$F_\kappa(B)\bar{H}(N_\sigma) = F_\kappa(\bar{B})\bar{H}(\bar{N}_\sigma) = \bar{H}(\bar{B}_s\bar{N}_\sigma).$$

But $\bar{B}_s \equiv \bar{B}^*, \text{ mod } \bar{\mathfrak{A}}_R$, so that $\bar{B}_s \bar{N}_\sigma \equiv \bar{B}^* \bar{N}_\sigma, \text{ mod } \bar{\mathfrak{A}}_R^{\sigma+1}$, and as $\bar{\mathfrak{S}}$ belongs to the σ^{th} power of the radical $\bar{\mathfrak{A}}_R$ of $\bar{\mathfrak{A}}$, we have

$$\bar{H}(\bar{B}_s \bar{N}_\sigma) = \bar{H}(\bar{B}^* \bar{N}_\sigma) = \bar{H}(\bar{B}^* \bar{N}_\sigma) = \bar{H}(B^* N_\sigma).$$

We may then view $\bar{\mathfrak{S}}$ as an $(\mathfrak{A}, \mathfrak{A})$ module of type (κ, λ) , related and belonging to $\mathfrak{N}^\sigma$, such that

$$B \circ \bar{H}(N_\sigma) = F_\kappa(B) \bar{H}(N_\sigma) = \bar{H}(B^* N_\sigma)$$

with a similar relation for right operators. Applying Lemma 1, we have that as a module related to $\mathfrak{N}^\sigma$, $\bar{\mathfrak{S}}$ is a linear combination of the elementary modules $\mathfrak{S}_{\kappa\lambda}^\mu$ of type (κ, λ) which belong to $\mathfrak{N}^\sigma$.¹¹ It follows, also, that a simple part $\bar{\mathfrak{C}}$ of $\bar{\mathfrak{A}}$ of type (κ, λ) which belongs to the σ^{th} power of the radical of $\bar{\mathfrak{A}}$ may, considered as a module related to $\mathfrak{N}^\sigma$, be expressed as a linear combination of the elementary modules $\mathfrak{S}_{\kappa\lambda}^\mu$ of type (κ, λ) which belong to $\mathfrak{N}^\sigma$.

If $\bar{\mathfrak{A}}^* = \bar{\mathfrak{A}}_s$ then the relation

$$F_\kappa(B) \bar{H}(A) = \bar{H}(B^* A)$$

holds for all elements A of $\mathfrak{A}$. Again applying Lemma 1 we obtain that $\bar{\mathfrak{S}}$ considered as a module related to $\mathfrak{A}$ itself is expressible as a linear combination of the elementary modules $\mathfrak{S}$ of type (κ, λ) which belong to $\mathfrak{N}^\tau, \tau \leq \sigma$.

If $\bar{\mathfrak{A}}^*$ is not identical to $\bar{\mathfrak{A}}_s$, that is, if $\bar{\mathfrak{A}}^*$ is not completely decomposed, then we can go over to an equivalent representation $\bar{\mathfrak{A}}$, such that $\bar{\mathfrak{A}}^* = \bar{\mathfrak{A}}_s$ where $\bar{\mathfrak{A}}^*, \bar{\mathfrak{A}}_s$ have the same meaning with regard to $\bar{\mathfrak{A}}$ as do $\bar{\mathfrak{A}}^*$ and $\bar{\mathfrak{A}}_s$ in regard to $\bar{\mathfrak{A}}$. We may then state:

THEOREM 8: *To any representation of $\mathfrak{A}$ there exists an equivalent representation whose simple parts, considered as modules related to $\mathfrak{A}$, are expressible as linear combinations of the elementary modules $\mathfrak{S}$ of $\mathfrak{A}$.*

5

It is easy now to discuss the center of $\mathfrak{A}$. We note that

THEOREM 9: *For a center element Z of $\mathfrak{A}$, $C_{ij}(Z) = 0$ if $\mathfrak{C}_{ij}$ is a simple part of mixed type, and $C_{ij}(Z)$ is of form $(c \cdot \delta_{mn})$, c an element of K , if $\mathfrak{C}_{ij}$ is of unmixed type.*

PROOF: Since an element Z of the center must commute with the elements $E_\kappa(mn)$ of $\mathfrak{A}$ ($\kappa = 1, 2, \dots, k; m, n = 1, 2, \dots, f_\kappa$) an easy calculation shows that in terms of the Cartan basis elements

$$Z = \sum_{r, r'} c^r(Z) E_\kappa(r1) B_{\kappa\kappa}^r E_\kappa(1r),$$

¹¹ If we do not take the constituents of $\bar{\mathfrak{A}}$ identical with constituents of $\mathfrak{A}$ then it is necessary to apply suitable left and right non-singular matrix multipliers to the $\mathfrak{S}_{\kappa\lambda}^\mu$ in the expression for $\bar{\mathfrak{S}}$.

in which the range for τ is determined by the s basis elements B^τ of unmixed type. It follows that $H^\mu(Z) = 0$ if $\mathfrak{S}^\mu$ is of mixed type, and

$$H^\mu(Z) = (c^\mu(Z)\delta_{ij})_{ij}$$

if $\mathfrak{S}^\mu$ is of unmixed type. Applying Theorem 4 we obtain the theorem.

It should be noted that this theorem can be obtained directly from Schur's Lemma and (2).

6

If an element is of type (κ, λ) we say that κ, λ are the type indices of the element. We say that two non-zero simple parts $\mathfrak{C}_{ij}, \mathfrak{C}_{pq}$, belong to the same block¹² if there exists a chain of non-zero parts such that any two neighboring parts have at least one type index in common. This relation is reflexive, symmetric, and transitive so that a separation of the non-zero simple parts into distinct disjoint classes is obtained.

THEOREM 10: *Each simple part in the i^{th} row of $\mathfrak{A}$ is either a 0-part or belongs to the same block as $\mathfrak{C}_{ii}$. Similarly, each simple part in the j^{th} column is either a 0-part or belongs to the same block as $\mathfrak{C}_{jj}$. If $\mathfrak{C}_{ii}, \mathfrak{C}_{jj}$ do not belong to the same block, then $\mathfrak{C}_{ij}$ is a 0-part.*

For if the simple part $\mathfrak{C}_{ij}$ is not a 0-part, then $\mathfrak{C}_{ii}, \mathfrak{C}_{ij}, \mathfrak{C}_{jj}$ is a proper chain and so $\mathfrak{C}_{ii}, \mathfrak{C}_{ij}, \mathfrak{C}_{jj}$ all belong to the same block.

In the same way we may classify the basis elements

$$(11) \quad B_{\kappa\lambda}^\mu, \quad (\mu = 1, 2, \dots, m; \kappa, \lambda = 1, 2, \dots, \text{or } k)$$

into blocks. We say that $B_{\kappa\lambda}^\mu, B_{\rho\sigma}^\gamma$ belong to the same block if there exists a chain of elements $B_{\alpha\beta}^\tau$ of the set (11) connecting $B_{\kappa\lambda}^\mu, B_{\rho\sigma}^\gamma$ such that any two neighboring elements in this chain have at least one type index in common. Let us denote by $\mathfrak{B}$ the set consisting of all the $B_{\rho\sigma}^\tau$ belonging to a given block and their associated elements of form $E_\rho(a1)B_{\rho\sigma}^\tau E_\sigma(1b)$. If A is any element of $\mathfrak{A}$, then $A \cdot E_\rho(a1)B_{\rho\sigma}^\tau E_\sigma(1b)$ is expressible in terms of $\mathfrak{B}$ again; the similar statement is true for the product $E_\rho(a1)B_{\rho\sigma}^\tau E_\sigma(1b) \cdot A$. It follows that the elements of $\mathfrak{B}$ form a basis of an invariant subalgebra of $\mathfrak{A}$: we denote this invariant subalgebra by $\mathfrak{A}^+$.

We can now prove

THEOREM 11: *By elementary transformations $\mathfrak{A}$ may be decomposed into the form*

$$(12) \quad \mathfrak{A} = \begin{pmatrix} \mathfrak{B}_1 & & & 0 \\ & \mathfrak{B}_2 & & \\ & & \ddots & \\ 0 & & & \mathfrak{B}_l \end{pmatrix}$$

where each $\mathfrak{B}_i$ contains all simple parts belonging to one block. The elements of $\mathfrak{A}$ which have only zeros in the parts $\mathfrak{B}_j, j \neq i$, form an invariant subalgebra $\mathfrak{A}_i$ of $\mathfrak{A}$,

¹² R. Brauer and C. Nesbitt, *On the Modular Representations of Finite Groups*, Univ. of Toronto Studies, Math. Series, No. 4 (1937); R. Brauer and C. Nesbitt, *On the Modular Character of Groups*, these Annals, Vol. 42 (1941), pp. 556-590; T. Nakayama, *Some Studies on Regular Representations, Induced Representations, and Modular Representations*, these Annals, Vol. 39 (1938), Theorem 5.

and $\mathfrak{A}$ is the direct sum of these invariant subalgebras $\mathfrak{A}_i$. The $\mathfrak{A}_i$ may not be decomposed into a direct sum of invariant subalgebras.

PROOF: If $\mathfrak{C}_{ii}$, $\mathfrak{C}_{i+1, i+1}$ do not belong to the same block, then by Theorem 10, $\mathfrak{C}_{i+1, 1}$ is a 0-part. It follows that by permuting the i^{th} row with the $(i+1)^{\text{st}}$ row and the i^{th} column with the $(i+1)^{\text{st}}$ column that we may reverse the positions of $\mathfrak{C}_{ii}$, $\mathfrak{C}_{i+1, i+1}$ in the main diagonal of $\mathfrak{A}$, but leave $\mathfrak{A}$ in reduced form. By successive transformations of this form we may bring together in an upper part $\mathfrak{B}_1$ of $\mathfrak{A}$ all $\mathfrak{C}_{ii}$ which belong to the same block as $\mathfrak{C}_{11}$. We then have

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{B}_1 & 0 \\ \mathfrak{D}_{21} & \mathfrak{D}_{22} \end{pmatrix}.$$

But since the simple parts $\mathfrak{C}_{jj}$ in $\mathfrak{B}_1$ and the $\mathfrak{C}_{ii}$ in $\mathfrak{D}_{22}$ belong now to different blocks, then the simple parts in $\mathfrak{D}_{21}$ must all be 0-parts, or, that is, $\mathfrak{D}_{21}$ is a 0-part. By continuing this process we obtain $\mathfrak{A}$ in the form (12).

To prove the remaining statements in the theorem we first remark that to each basis element $B_{\kappa\lambda}^\mu$ there corresponds at least one non-zero simple part $\mathfrak{C}_{ij}$ of type (κ, λ) . For, by definition, $B_{\kappa\lambda}^\mu$ is a matrix of $\mathfrak{A}$ having non-zero coefficients only in the upper left corners of simple parts of type (κ, λ) .

Let now $B_{\rho\sigma}^\tau$ be an element of the set $\mathfrak{B}$ of Cartan basis elements which belong to a given block. If the simple parts of type (ρ, σ) are in part $\mathfrak{B}_i$ of $\mathfrak{A}$, then by the above remark $B_{\rho\sigma}^\tau$ is an element of the invariant subalgebra $\mathfrak{A}_i$. If $B_{\mu\nu}^\gamma$ is also from the set $\mathfrak{B}$, that is, if $B_{\mu\nu}^\gamma$ belongs to the same block as $B_{\rho\sigma}^\tau$, then the simple parts of type (μ, ν) belong to the same block as simple parts of type (ρ, σ) . For a simple part of type (ρ, σ) may be connected by a chain of non-zero simple parts to a simple part of type (μ, ν) , the members of this chain corresponding to the members of the chain connecting $B_{\rho\sigma}^\tau$, $B_{\mu\nu}^\gamma$. Then the simple parts of type (μ, ν) are in the part $\mathfrak{B}_i$ and so $B_{\mu\nu}^\gamma$ also is an element of $\mathfrak{A}_i$. It follows that the invariant subalgebra $\mathfrak{A}^+$, generated by the elements of $\mathfrak{B}$, coincides with $\mathfrak{A}_i$. We now have at our disposal a correspondence between invariant subalgebras determined by blocks of Cartan basis elements on the one hand, and by blocks of simple parts on the other.

Now let

$$(13) \quad \mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 + \cdots + \mathfrak{A}_q$$

be a direct decomposition of $\mathfrak{A}$ into invariant subalgebras which may not be directly decomposed, and let $\bar{A}_j$ be an element of $\mathfrak{A}_j$. If in the expression

$$\bar{A}_j = \sum h_{ab}^\mu(\bar{A}_j) E_\kappa(a1) B_{\kappa\lambda}^\mu E_\lambda(1b)$$

for $\bar{A}_j$ in terms of the Cartan basis elements, $h_{ab}^\mu(\bar{A}_j) \neq 0$, then since

$$E_\kappa(1a) \bar{A}_j E_\lambda(1b) = h_{ab}^\mu(\bar{A}_j) B_{\kappa\lambda}^\mu$$

is in $\mathfrak{A}_j$, so also is $B_{\kappa\lambda}^\mu$. In this way we can determine the direct summand of the decomposition (13) to which each element $B_{\kappa\lambda}^\mu$ belongs. In particular, we may determine the summands to which the elements $E_\kappa(11)$, $(\kappa = 1, 2, \dots, k)$ belong. If $B_{\kappa\lambda}^\mu$ belongs to $\mathfrak{A}_j$ then $E_\kappa(11)$, $E_\lambda(11)$ must also belong to $\mathfrak{A}_j$, since

$$E_\kappa(11) B_{\kappa\lambda}^\mu = B_{\kappa\lambda}^\mu E_\lambda(11) = B_{\kappa\lambda}^\mu$$

and if, for instance, $E_\kappa(11)$ did not belong to the same summand as $B_{\kappa\lambda}^\mu$, the product $E_\kappa(11)B_{\kappa\lambda}^\mu$ would be zero.

Suppose now that the element $B_{\rho\sigma}^\tau$ of the set $\mathfrak{B}$ is in $\mathfrak{A}_j$ but that there are elements of $\mathfrak{B}$ which are not in $\mathfrak{A}_j$. Then we may choose an element $B_{\mu\nu}^\gamma$ of $\mathfrak{B}$ which is not in $\mathfrak{A}_j$, but such that in the chain joining $B_{\rho\sigma}^\tau$ to $B_{\mu\nu}^\gamma$, $B_{\mu\nu}^\gamma$ is the only member which does not belong to $\mathfrak{A}_j$. The member preceding $B_{\mu\nu}^\gamma$ in this chain has either the type index μ or ν ; let this member be $B_{\mu\lambda}^\delta$. Then by the above, $E_\mu(11)$ is in $\mathfrak{A}_j$, since $B_{\mu\lambda}^\delta$ is in $\mathfrak{A}_j$; on the other hand $E_\mu(11)$ must be in the same summand as $B_{\mu\nu}^\gamma$, which gives a contradiction. Thus all elements of the set $\mathfrak{B}$ are in $\mathfrak{A}_j$. Similarly, if any member of a second set $\mathfrak{B}_1$ of Cartan basis elements which all belong to one block is in $\mathfrak{A}_j$, then all members of the set $\mathfrak{B}_1$ are in $\mathfrak{A}_j$. But then the invariant subalgebras $\mathfrak{A}^+$, $\mathfrak{A}_1^+$ which have as bases the sets $\mathfrak{B}$, $\mathfrak{B}_1$, respectively, would be direct summands of $\mathfrak{A}_j$ contrary to our assumption that $\mathfrak{A}_j$ is directly indecomposable. We now have $\mathfrak{A}_i = \mathfrak{A}^+ = \mathfrak{A}_j$ which completes our proof.

We suppose now that each $\mathfrak{B}_i$ has been split into its indecomposable parts¹³ and that these are in reduced form. Let $U_1^i, U_2^i, \dots, U_n^i$ be the indecomposable parts of $\mathfrak{B}_i$. We now classify the U^i by the following relation: we say that U_ρ^i, U_σ^i belong to the same block if there exists a chain¹⁴

$$(14) \quad U_\rho^i, U_\alpha^i, \dots, U_\beta^i, U_\sigma^i$$

such that any two neighboring parts of the chain (14) have at least one irreducible constituent in common. We will show that all indecomposable parts U_ρ^i ($\rho = 1, 2, \dots, n$) of $\mathfrak{B}_i$ belong to one block. Let us take together all U_ρ^i that are connected with U_1^i by such a chain and suppose U_α^i does not belong to this set. Then the simple parts of U_α^i cannot be connected by a proper chain of non-zero simple parts of U_1^i , contrary to our provision that simple parts of $\mathfrak{B}_i$ all belong to one block.

Since a matrix commuting with an indecomposable part U_ρ^i can have just one distinct characteristic root¹⁵ and since, as we have seen, all U_ρ^i of $\mathfrak{B}_i$ have at least one irreducible constituent in common, then for an element Z of the center of $\mathfrak{A}$ the matrix $B_i(Z)$ of $\mathfrak{B}_i$ corresponding to Z has just one distinct characteristic root.

We have then proved

THEOREM 12: *The indecomposable constituent of a part $\mathfrak{B}_i$ may be characterized as belonging to a block. It follows that for a center element Z , the corresponding matrix $B_i(Z)$ has only one distinct characteristic root.*

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¹³ For definitions of indecomposable parts, see B.N., or N.R.

¹⁴ R. Brauer and C. Nesbitt, *On the Modular Representations of Groups of Finite Order*, Univ. of Toronto Studies, Math. Series, No. 4 (1937), p. 14.

¹⁵ R. Brauer and I. Schur, *Zum Irreduzibilitätsbegriff in der Theorie der Gruppen linearer homogener Substitutionen*, Sitzber. der Preuss. Akad. der Wiss., Phys-Math. Klasse, Berlin, XIV (1930).

